see:
Galtier
(Kulsrud)
Frisch

MHD Turbulence I

i.) Key Features of Hydro. Turbulence

ii.) Basic Facts re: MHD Turbulence

iii.) Phenomenology ala' Kraichnan / Iroshnikov

iv.) Critical Balance and Phenomenology
ala' Goldreich-[redacted] Sridhar

v.) 4/5 Law Analogue + Further.

i.) Key Features of Hydro. Turbulence (3D)
(Heavily based on experiment — C_D vs Re)

- chaotic state at high Re
characterized by broad self-similar range and nonlinear transfer (flux cnk.)

- dissipates energy; dissipation rate
indep. Re, as $Re \to \infty$ (but $\nu \neq 0$).

i.e. $\langle \tilde{f} \tilde{v} \rangle = \epsilon = \nu \langle (\nabla v)^2 \rangle$
$= \nu \langle \omega^2 \rangle$

$\Rightarrow$ singularity formation $l_\nu \sim 1/\sqrt{\nu}$

- $l_d < l < l_0$

$\frac{1}{\tau(l)} \to$ scale similarity constrained by Galilean invariance.

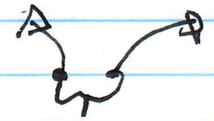
$\epsilon = \frac{v(l)^2}{\tau(l)} = \frac{v(l)^3}{l}$, etc.

$l_d = \nu^{3/4} / \epsilon^{1/4}$

"turbulent flow is rough"

$\Rightarrow E(k) \cong \epsilon^{2/3} k^{-5/3}$

- alternatively



$\delta\ell$

$$\frac{d\,\delta\ell}{dt} = v(\delta\ell)$$

$\delta\ell^2 \sim \epsilon^2 t^3$ $\Rightarrow$ Super-diffusive separation

- Some Things You Can Trust:

a.) Karman-Howarth Eqn. (Modulo usual)

$$\partial_t \left\langle \frac{u_i u_i'}{2} \right\rangle = \frac{1}{4} \nabla_{\ell} \cdot \langle |\underbrace{\delta u}_{u(x+\ell)-u(x)} \cdot \delta u|\, \delta u \rangle + 2\nu\, \partial^2_{\ell_n, \ell_n} \left\langle \frac{u_i u_i'}{2} \right\rangle$$

$u_i = u(x)$

$u_i' = u(x')$

$x' = x + \ell$

with external forcing:

$$\partial_t \left\langle \frac{u_i u_i'}{2} \right\rangle = \frac{1}{4} \nabla_{\ell} \cdot \langle (\delta u_i)^2 \delta u \rangle + 2\nu\, \partial^2_{\ell_n, \ell_n} \left\langle \frac{u_i u_i'}{2} \right\rangle + F$$

$\Rightarrow$ scale energetics balance

b.) 4/5 Law

Balancing external stirring with energy dissipation (must for steady state):

$$\cancelto{0\ (\mathrm{SS})}{\partial_t \left\langle \frac{u_i u_i'}{2} \right\rangle} = \frac{1}{4} \nabla_\ell \cdot \langle (\delta u_i)^2 \delta \underline{u} \rangle$$

$$+ \underbrace{2\nu \, \partial_{\ell_k} \partial_{\ell_k} \left\langle \frac{u_i u_i'}{2} \right\rangle}_{\text{i.r.}} + \epsilon$$

in inertial range:

$$\langle (\delta u_\ell)^3 \rangle = -\frac{4}{5} \epsilon \ell \quad \text{— decay}$$

i.e. energy transfer at scale ℓ (arbitrary in inertial range)

$$\boxed{\frac{\langle (\delta u_\ell)^3 \rangle}{\ell} = -\frac{4}{5} \epsilon}$$

proportional (with 4/5 factor) to dissipation rate b.) MHD (3D, Incompressible)

① — often wave strong, slowly varying external magnetic field

so $\rho(\partial_t \underline{V} + \underline{V}\cdot\nabla\underline{V}) = -\nabla p^* + \frac{\underline{B}\cdot\nabla\underline{B}}{4\pi} + \nu\nabla^2\underline{V} + \tilde{f}$

$\partial_t \underline{B} + \underline{V}\cdot\nabla\underline{B} = \underline{B}\cdot\nabla\underline{V} + \eta\nabla^2\underline{B}$ → general → need not $\underline{B_0}$

$\nabla\cdot\underline{B} = 0$

Now here can (for $\underline{B_0}$) use reduced MHD:

$$\rho(\partial_t \nabla_\perp^2\phi + \underline{V}_\perp\cdot\nabla_\perp\nabla^2\phi) = B_0\partial_z\nabla^2 A + \nabla_\perp A\times\hat{z}\cdot\nabla_\perp\nabla^2 A + \tilde{f} + \nu\nabla^2\nabla^2\phi$$

⊥ → emphasize

$$\partial_t A + \underline{V}_\perp\cdot\nabla_\perp A = B_0\partial_z\phi + \eta\nabla^2 A$$

→ anisotropic turbulence, if B_0 strong. (2D + SAW in z)

→ wave component i.e. MHD turbulence is gas of waves (Alfvenic)

⇒ expect use methodology of wave turbulence.

$\underline{B_0}$ strong → wave turbulence.

(A) → nonlinear transfer in $\perp$, i.e. $k_\perp$ (3D turbulence has coupling also 2D)

→ what controls triad coherence?

⇒ Alfvenic transit (over scale)

From HW:

(2) Nonlinear interaction exclusively via counter-propagating Elsasser populations

i.e. $\underline{Z}^{\pm} = \underline{V} \pm \underline{b}$ $\qquad$ $\underline{b} \equiv$ pert. $\tilde{B}$

2 MHD eqns ⇒

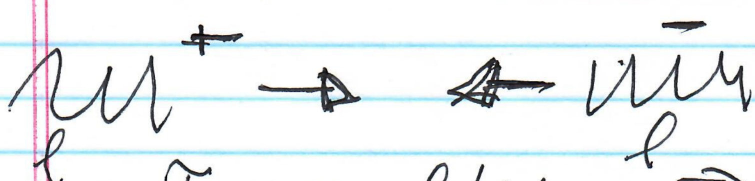
$$\partial_t \underline{Z}^{\pm} \mp \underline{b}_0 \cdot \underline{\nabla} Z^{\pm} + Z^{\mp} \cdot \nabla Z^{\pm} = -\nabla p^*$$

$$+ \nu_+ \nabla^2 \underline{Z}^{\pm} + \nu_- \nabla^2 Z^{\mp}$$

$$\nu_\pm = \frac{\nu \pm \eta}{2}$$

so NL coupling exclusively via $\underline{Z}^{\mp} \cdot \underline{\nabla} \underline{Z}^{\pm}$

so NL transfer $\qquad$ interaction diff'nt populations



$\tau_{int} \sim l / V_A$ ⇒ coherence time.

i.e. counter-propagating packets' interaction time limited to l/V_A.

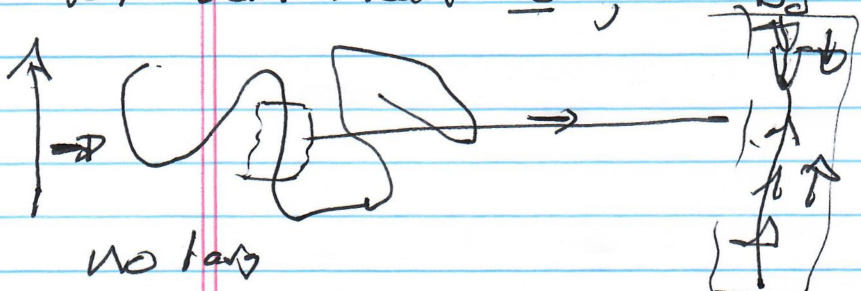
i.e. saves us from renormalization in absence of dispersion.

Classic

so, some Phenomenology: (Iroshnikov, Kraichnan) ('64, '65)

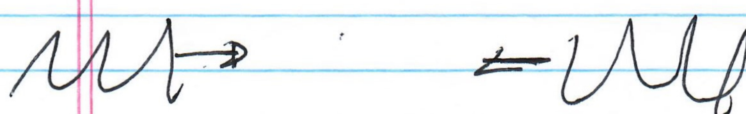
→ nonlinear transfer — ~ isotropic turbulence

For weak mean $\underline{B_0}$, $b_0 \sim b_{rms} > |B_0|$



locally, ∃ a $\underline{b_0}$

→ so, nonlinear scattering from "collisions" of counter-propagating packets



[universally here, ≡ counter Els. populations locality?]

so (modulo 4/5 !) — TBD.

$$\to \boxed{\epsilon \sim \frac{Z_\ell^2}{\tilde{\tau}_{T\ell}}}$$

$\ell \equiv$ scale

→ Alfven wave cascade

→ const. energy thru-put rat

Now Alfvenic transit time on scale ℓ, in b_0 is

$$\boxed{\tau_A = \ell / b_0}$$

For transfer:

- transfer by wave scattering $\Leftrightarrow$ random walk of amplitude / interns
- define τ_{Tr} by:

$$\langle \delta z^2(\tau_{Tr}) \rangle \sim \langle z^2 \rangle$$

τ_{Tr} sufficient # kicks to make

i.e. randomization of scattering in amplitude

$$\underline{\delta z_{rms} \sim z}$$

so

$$z_\ell(t+\tau_A) = z_\ell(t) + \tau_A \partial_t z_\ell$$

kick in 1 Alfven transit time

$$\delta z_\ell \simeq \tau_A \frac{z_\ell^2}{\ell} \rightarrow \underline{\text{kick}} \text{ in } \tau_A$$

so $\frac{z_\ell^2}{\ell} \rightarrow$ kick.

$\Rightarrow$

$$\langle \delta z_\ell^2 \rangle = \left(\tau_A \frac{z_\ell^2}{\ell}\right)^2 \frac{t}{\tau_A} \rightarrow \boxed{\text{accumulated kicks}}$$

$$= \underbrace{\left(\frac{z_\ell^2}{\ell}\right)^2 \tau_A}\, t = D t$$

thus:

$$\langle \delta z_\ell^2 \rangle \sim \langle z_\ell^2 \rangle$$

$$\Rightarrow z_\ell^2 \simeq \tau_A^2 \left(\frac{z_\ell^2}{\ell^2}\right)^2 \frac{\tau_{Tr}}{\tau_A}$$

$$\therefore \boxed{\frac{1}{\tau_{Tr}} \simeq \frac{z_\ell^2}{\ell^2} \tau_A} \rightarrow \underline{\text{determines transfer rate}}.$$

so $\frac{1}{\tau_{Tr}} \sim \frac{Z_\ell^2}{\ell^2}\frac{\ell}{b_0}$

$$\sim \frac{Z_\ell^2}{\ell\, b_0}$$

Thus:

$$\epsilon \simeq \frac{Z_\ell^2}{\tau_{Tr}} \simeq \frac{Z_\ell^2 Z_\ell^2}{\ell\, b_0}$$ const. rate balance

$$\boxed{Z_\ell^2 \sim \sqrt{b_0 \epsilon}\, \ell^{1/2}}$$ $Z_\ell \sim (b_0 \epsilon)^{1/4} \ell^{1/4}$

$$\boxed{E(k) \simeq \sqrt{\epsilon b_0}\, k^{-3/2}}$$ Kraichnan spectrum!

N.B. – as work with Z,
$E_M \sim E_K$.
(equal Elsasser populations)

– $1/\tau_{Tr} \simeq \frac{Z_\ell^2}{\ell^2}\frac{\ell}{b_0} \simeq \underbrace{\frac{\tau_A}{\tau_{eddy}}}\frac{1}{\tau_{eddy}}$

compare K41: $1/\tau_{Tr} \sim \frac{1}{\tau_{eddy}}$

Transfer rate reduced by factor $\underline{\tau_A/\tau_{eddy}}$!

Also, recall for weak wave turbulence:

$$C\{N\} = \sum_{k'} |V|^2 N_{k'} N_k \tau_{Tc}$$ → general structure of collision integral

τ_{Tc} ↓ (triad) coherence time → interaction

so
$$\boxed{\frac{1}{\tau_{Tr}} \simeq \sum_{k'} |V|^2 N_{k'} \tau_{Tc}}$$

but $N_k \to Z_\ell^2$ (intensity) (assumes local transfer)

$$|V|^2 \sim |\nabla^2| \sim \frac{1}{\ell^2} \to \frac{1}{\ell_\perp^2}$$

$$\tau_{Tc} \sim \tau_A \to \text{triad coherence}$$

τ_A ↓ Alfvenic packet transit time

$$1/\tau_{Tr} \sim \frac{Z_\ell^2}{\ell^2} \tau_A \Leftrightarrow \text{W.T.T.}$$

fundamentally wave scattering process!

- 'triad' → 2AWs + cell. → akin NLLD.

$\underline{q} = \underline{k} + \underline{k}'$ (cargo)

$\omega_{\underline{q}} = \omega_{\underline{k}} + \omega_{\underline{k}'}$

$\rightarrow 0 = k_{\parallel} V_A - k'_{\parallel} V_A$

- Alfven waves non-dispersive, but coherence time controlled by packet propagation

⇒ prevents negative spectra, etc. (coherence time not so large)

- Strong B_0. (explicit)

Some simple observations:

- turbulence clearly anisotropic

- nonlinear transfer in $k_\perp$.

So, consider weak wave turbulence!

Consider: $k_\perp \to 1/l_\perp$

$k_\parallel$

$k_\perp > k_\parallel$ so

so $$\epsilon \sim \frac{(Z(l_\perp)^2)(Z(l_\perp)^2)}{l_\perp^2 \, |k_\parallel V_A|}$$

$\Delta k_\parallel V_A \to$ coh. time

i.e. tacitly $\Delta |k_\parallel V_A| \sim |k_\parallel V_A|$

$$z(\ell_\perp)^2 \cong (\epsilon |k_\parallel V_A|)^{1/2}\, \ell_\perp$$

$\Rightarrow$

$$E(k_\perp) \cong (\epsilon\, |k_\parallel V_A|)^{1/2} / k_\perp^2$$

sharp
cut. restricted

and accounting for DOS in $k_\parallel$:

$$\boxed{E(k_\perp, k_\parallel) \sim [\epsilon\, V_A]^{1/2} / k_\parallel^{1/2} k_\perp^2}$$

weak wave turbulence spectrum

- strongly anisotropic
- $k_\parallel$ frozen.

Now, $\quad z(\ell_\perp) \sim \delta B(\ell_\perp) \sim (\epsilon |k_\parallel V_A|)^{1/4}\, \ell_\perp^{1/2}$

$$\omega \sim k_\parallel V_A$$

recall: $\quad \nabla_\parallel = \partial_z + \frac{\delta B_\perp}{B_0}\cdot \nabla_\perp$

Kubo #

$$\boxed{k_\parallel \sim \frac{\delta B_\perp}{B_0}\nabla_\perp \Big/ \partial_z \sim \frac{l_{ac}\,\delta B_\perp / B_0}{\Delta_\perp}}$$

(see 235)

Now, $\quad \frac{\delta B_\perp}{\ell_\perp} \sim (\epsilon\, k_\parallel V_A)^{1/4} / \ell_\perp^{1/2}$

kicks in coherence length

⇒ For W.T.T., expect:

$k u < 1$ (diffusive picture)

but,

$k u$ rises as $l_\perp$ drops

i.e. $k u \uparrow$ as progress thru $l_\perp$ cascade! — ?? what happens

⇒ Begs the question:

How high can $k u$ go and still retain physics of Alfven wave cascade?

enter the:

$$\boxed{\text{Critical Balance Conjecture}}$$

— Goldreich–Sridhar (1995)
(cf also Kadomtsev–Pogutse, 1978)

⇒ $\boxed{\text{MHD inertial range in strong field will sit at } k u \sim 1}$

i.e. $\delta B_\perp \cdot \nabla_\perp \sim B_0 \nabla_\parallel$

i.e. transit time sets bound on cnt. strength

$$\frac{z(l_\perp)}{l_\perp} \simeq k_\parallel V_A$$

$\frac{\tau_A}{\tau_{eddy}} \to 1$, $\frac{\tau_{tr}}{\tau_{eddy}} \to$

⋮

but: $$\epsilon \sim \frac{(z(\ell_\perp))^2 (z(\ell_\perp))^2}{\ell_\perp^2} \frac{1}{k_\parallel V_A}$$

$$\rightarrow \frac{(z(\ell_\perp))^2 (z(\ell_\perp))^2}{\ell_\perp^2 \, \frac{z(\ell_\perp)}{\ell_\perp}}$$

$$= \frac{z(\ell_\perp)^3}{\ell_\perp}$$

$$\Rightarrow \quad z(\ell_\perp) \sim (\epsilon \ell_\perp)^{1/3}$$

$$\boxed{E(k_\perp, k_\parallel) \sim \epsilon^{2/3} k_\perp^{-5/3}}$$

G-S spectrum

→ back to K41, but diff'nt physics

→ fits data

and anisotropy:

$k_\parallel \sim 1$, $\quad B_0 \cdot \nabla \sim \tilde{B}_\perp \cdot \nabla_\perp$

$$B_0 k_\parallel \sim \epsilon^{1/3} \frac{\ell_\perp^{1/3}}{\ell_\perp}$$

$k_\parallel$ vs $k_\perp$ relation

$$\boxed{k_\parallel \sim \frac{\epsilon^{1/3}}{B_0} k_\perp^{+2/3}}$$

specifies "G-S cone" in $\underline{k}$ space

i.e. $\quad k_\parallel \ll k_\perp \quad$ → $\boxed{\text{cascade develops preferentially in perp.}}$

– Why Believe

→ analogue of 4/5 Law (Pouquet, Politano)

$$-\frac{4}{3}\epsilon^T \ell = \langle (\delta v\cdot\delta v + \delta b\cdot\delta b)\,\delta v_\ell \rangle$$

↳ mom bal → form $\dot{E}_K$

$$-2\langle (\delta v\cdot\delta b)\, b_\ell \rangle$$

↓ total energy cascade

↓ induction → form $\dot{E}_M$

~ reflects flip-flop in energy between channels

<u>or</u>

$$-\frac{4}{3}\epsilon^{\pm}\ell = \langle (\delta z^{\pm}\cdot\delta z^{\pm})\,\delta z_\ell^{\mp} \rangle$$

no dissipn for 1 stream only.

→ Why? → relate induced E-field.